

A Corollary for Nonsmooth Systems

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Abstract—In this note, two generalized corollaries to the LaSalle-Yoshizawa Theorem are presented for nonautonomous systems described by nonlinear differential equations with discontinuous right-hand sides. Lyapunov-based analysis methods are developed using differential inclusions to achieve asymptotic convergence when the candidate Lyapunov derivative is upper bounded by a negative semi-definite function.

I. INTRODUCTION

For continuous systems, stability techniques such as the LaSalle-Yoshizawa Theorem provide a convenient analysis tool when the candidate Lyapunov function derivative is upper bounded by a negative semi-definite function. However, adapting the LaSalle-Yoshizawa Theorem to systems where the time derivative of the system states are not locally Lipschitz remains an open problem. The concept of utilizing the LaSalle-Yoshizawa Theorem for nonsmooth systems was introduced in [1] as a remark, but no formal proof was provided.

In this note, we consider Filippov solutions for nonautonomous nonlinear systems with right-hand side discontinuities¹ utilizing Lipschitz continuous and regular Lyapunov functions whose time derivatives (in the sense of Filippov) can be upper bounded by negative semi-definite functions.

II. PRELIMINARIES

Consider the system

$$\dot{x} = f(x, t) \quad (1)$$

where $x(t) \in D \subset \mathbb{R}^n$ denotes the state vector, and $f : D \times [0, \infty) \rightarrow \mathbb{R}^n$ is a Lebesgue measurable and essentially locally bounded [2] function. As is standard in literature [3], existence and uniqueness of the continuous solution $x(t)$ are provided under the condition that the function f is Lipschitz continuous. However, if f contains a discontinuity at any point in D , then a solution to (1) may not exist in the classical sense. Thus, it is necessary to redefine the concept of a solution. Utilizing differential inclusions, the value of a generalized solution (e.g., Filippov [4] or Krasovskii [5] solutions) at a certain point can be found by interpreting the behavior of its derivative at nearby points. Generalized solutions will be close to the trajectories of the actual system since they are a limit of solutions of ordinary differential equations with a continuous right-hand side [6]. While there exists a Filippov solution for any arbitrary initial condition $x(t_0) \in D$, the solution is generally not unique [4], [7].

Definition 1. (Filippov Solution) [4] A function $x(t)$ is called a solution of (1) on the interval $[0, \infty)$ if $x(t)$ is

¹Throughout the subsequent presentation, a discontinuous right-hand side will refer to being discontinuous in x , and continuous in t .

absolutely continuous and for almost all $t \in [0, \infty)$,

$$\dot{x} \in K[f](x, t)$$

where

$$K[f](x, t) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu N=0} \overline{\text{co}}f(B(x, \delta) \setminus N, t), \quad (2)$$

$\bigcap_{\mu N=0}$ denotes the intersection over sets N of Lebesgue measure zero, $\overline{\text{co}}$ denotes convex closure, and $B(x, \delta) = \{v \in \mathbb{R}^n \mid \|x - v\| < \delta\}$.

To facilitate the main results, three definitions are provided.

Definition 2. (Directional Derivative) [8] Given a function $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$, the right directional derivative of f at $x \in \mathbb{R}^m$ in the direction of $v \in \mathbb{R}^m$ is defined as

$$f'(x, v) = \lim_{t \rightarrow 0^+} \frac{f(x + tv) - f(x)}{t}.$$

Additionally, the generalized directional derivative of f at x in the direction of v is defined as

$$f^o(x, v) = \lim_{y \rightarrow x} \sup_{t \rightarrow 0^+} \frac{f(y + tv) - f(y)}{t}.$$

Definition 3. (Regular Function) A function $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is said to be regular at $x \in \mathbb{R}^m$ if for all $v \in \mathbb{R}^m$, the right directional derivative of f at x in the direction of v exists and $f'(x, v) = f^o(x, v)$.²

Definition 4. (Clarke's Generalized Gradient) [10] For a function $V : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ that is locally Lipschitz in (x, t) , define the generalized gradient of V at (x, t) by

$$\partial V(x, t) = \overline{\text{co}}\{\lim \nabla V(x, t) \mid (x_i, t_i) \rightarrow (x, t), (x_i, t_i) \notin \Omega_V\}$$

where Ω_V is the set of measure zero where the gradient of V is not defined.

The following lemma provides a method for computing the time derivative of a regular function $V(x, t)$ using Clarke's generalized gradient [10] and $K[f](x, t)$, from (2), along the solution trajectories of (1).

Lemma 1. (Chain Rule) [11], [12] Let $x(t)$ be a Filippov solution of (1) and $V : D \times [0, \infty) \rightarrow \mathbb{R}$ be a locally Lipschitz, regular function. Then $V(x(t), t)$ is absolutely continuous, $\frac{d}{dt}V(x(t), t)$ exists almost everywhere (a.e.), i.e., for almost all $t \in [0, \infty)$, and $\dot{V}(x(t), t) \stackrel{\text{a.e.}}{\in} \dot{V}(x(t), t)$, where

$$\dot{V}(x, t) \triangleq \bigcap_{\xi \in \partial V(x, t)} \xi^T \begin{pmatrix} K[f](x, t) \\ 1 \end{pmatrix}.$$

Remark 1. Throughout the subsequent discussion, for brevity of notation, let a.e. refer to almost all $t \in [0, \infty)$.

²Note that any C^1 continuous function is regular and the sum of regular functions is regular [9].

III. MAIN RESULT

For the system described in (1) with a continuous right-hand side, existing Lyapunov theory can be used to examine the stability of the closed-loop system using continuous techniques such as those described in [3]. However, these theorems must be altered for the set-valued map $\dot{V}(x(t), t)$ for systems with right-hand sides which are not Lipschitz continuous [6], [12], [13]. Lyapunov analysis for nonsmooth systems is analogous to the analysis used for continuous systems. The differences are that differential equations are replaced with inclusions, gradients are replaced with generalized gradients, and points are replaced with sets throughout the analysis. The following presentation and subsequent proofs demonstrate how the LaSalle-Yoshizawa Theorem can be adapted for such systems.

The following auxiliary lemma from [11] and Barbalat's Lemma are provided to facilitate the proofs of the nonsmooth LaSalle-Yoshizawa Corollaries.

Lemma 2. [11] Let $x(t)$ be any Filippov solution to the system in (1) and $V : \mathcal{D} \times [0, \infty) \rightarrow \mathbb{R}$ be a locally Lipschitz, regular function. If $\dot{V}(x(t), t) \stackrel{a.e.}{\leq} 0$, then $V(x(t), t) \leq V(x(t_0), t_0) \forall t > t_0$.

Proof: For the sake of contradiction, let there exist some $t > t_0$ such that $V(x(t), t) > V(x(t), t_0)$. Then,

$$\int_{t_0}^t \dot{V}(x(\sigma), \sigma) d\sigma = V(x(t), t) - V(x(t), t_0) > 0.$$

It follows that $\dot{V}(x(t), t) > 0$ on a set of positive Lebesgue measure, which contradicts that $\dot{V}(x(t), t) \leq 0$, a.e. ■

Lemma 3. (Barbalat's Lemma) [3] Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a uniformly continuous function on $[0, \infty)$. Suppose that $\lim_{t \rightarrow \infty} \int_{t_0}^t \phi(\tau) d\tau$ exists and is finite. Then,

$$\phi(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Based on Lemmas 2 and 3, a nonsmooth corollary to the LaSalle-Yoshizawa Theorem (c.f., [14, Theorem 8.4] and [15, Theorem A.8]) is provided in Corollary 1.

Corollary 1. For the system given in (1), let $\mathcal{D} \subset \mathbb{R}^n$ be a domain containing $x = 0$ and suppose f is Lebesgue measurable and essentially locally bounded on $\mathcal{D} \times [0, \infty)$. Furthermore, suppose $f(0, t)$ is uniformly bounded for all $t \geq 0$. Let $V : \mathcal{D} \times [0, \infty) \rightarrow \mathbb{R}$ be continuously differentiable in x , locally Lipschitz in t , and regular such that

$$W_1(x) \leq V(x(t), t) \leq W_2(x) \quad \forall t \geq 0, \quad \forall x \in \mathcal{D}, \quad (3)$$

$$\dot{V}(x(t), t) \stackrel{a.e.}{\leq} -W(x) \quad (4)$$

where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions, $W(x)$ is a continuous positive semi-definite function on \mathcal{D} , choose $r > 0$ and $c > 0$ such that $B_r \subset \mathcal{D}$ and $c < \min_{\|x\|=r} W_1(x)$ and $x(t)$ is a Filippov solution to (1) where $x(t_0) \in \{x \in B_r \mid W_2(x) \leq c\}$. Then $x(t)$ is bounded and satisfies

$$W(x(t)) \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (5)$$

Proof: Since $x(t)$ is a Filippov solution to (1), $\{x \in B_r \mid W_1(x) \leq c\}$ is in the interior of B_r . Define a time-dependent set $\Omega_{t,c}$ by

$$\Omega_{t,c} = \{x \in B_r \mid V(x, t) \leq c\}.$$

From (3), the set $\Omega_{t,c}$ contains $\{x \in B_r \mid W_2(x) \leq c\}$ since

$$W_2(x) \leq c \Rightarrow V(x, t) \leq c.$$

On the other hand, $\Omega_{t,c}$ is a subset of $\{x \in B_r \mid W_1(x) \leq c\}$ since

$$V(x, t) \leq c \Rightarrow W_1(x) \leq c.$$

Thus,

$$\{x \in B_r \mid W_2(x) \leq c\} \subset \Omega_{t,c},$$

$$\Omega_{t,c} \subset \{x \in B_r \mid W_1(x) \leq c\} \subset B_r \subset \mathcal{D}.$$

Based on (4), $\dot{V}(x(t), t) \stackrel{a.e.}{\leq} 0$, hence, $V(x(t), t)$ is non-increasing from Lemma 2. For any $t_0 \geq 0$ and any $x(t_0) \in \Omega_{t_0,c}$, the solution starting at $(x(t_0), t_0)$ stays in $\Omega_{t,c}$ for every $t \geq t_0$. Therefore, any solution starting in $\{x \in B_r \mid W_2(x) \leq c\}$ stays in $\Omega_{t,c}$, and consequently in $\{x \in B_r \mid W_1(x) \leq c\}$, for all future time. Hence, the Filippov solution $x(t)$ is bounded such that $\|x(t)\| < r$, $\forall t \geq t_0$.

From Lemma 2, $V(x(t), t)$ is also bounded such that $V(x(t), t) \leq V(x(t_0), t_0)$. Since $\dot{V}(x(t), t)$ is Lebesgue measurable from (4),

$$\int_{t_0}^t W(x(\tau)) d\tau \leq - \int_{t_0}^t \dot{V}(x(\tau), \tau) d\tau, \quad (6)$$

$$-\int_{t_0}^t \dot{V}(x(\tau), \tau) d\tau = V(x(t_0), t_0) - V(x(t), t) \leq V(x(t_0), t_0).$$

Therefore, $\int_{t_0}^t W(x(\tau)) d\tau$ is bounded $\forall t > t_0$. Existence of $\lim_{t \rightarrow \infty} \int_{t_0}^t W(x(\tau)) d\tau$ is guaranteed since the left-hand side of (6) is monotonically nondecreasing (based on the definition of $W(x)$) and bounded above. Since every absolutely continuous function is uniformly continuous, $x(t)$ is uniformly continuous. Because $W(x)$ is continuous in x , and x is on the compact set B_r , $W(x(t))$ is uniformly continuous in t on $(t_0, \infty]$. Therefore, by Lemma 3,

$$W(x(t)) \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (7)$$

■

Remark 2. From Def. 1, $K[f](x, t)$ is an upper semi-continuous, nonempty, compact and convex valued map. While existence of a Filippov solution for any arbitrary initial condition $x(t_0) \in \mathcal{D}$ is provided by the definition, generally speaking, the solution is non-unique [4], [7].

Note that Corollary 1 establishes (7) for a specific $x(t)$. Under the stronger condition that³ $\dot{\tilde{V}}(x, t) \leq W(x) \forall x \in \mathcal{D}$, it is possible to show that (7) holds for all Filippov solutions of (1). The next corollary is presented to illustrate this point.

³The inequality $\dot{\tilde{V}}(x, t) \leq W(x)$ is used to indicate that every element of the set $\dot{\tilde{V}}(x, t)$ is less than or equal to the scalar $W(x)$.

Corollary 2. For the system given in (1), let $\mathcal{D} \subset \mathbb{R}^n$ be a domain containing $x = 0$ and suppose f is Lebesgue measurable and essentially locally bounded on $\mathcal{D} \times [0, \infty)$. Furthermore, suppose $f(0, t)$ is uniformly bounded for all $t \geq 0$. Let $V : \mathcal{D} \times [0, \infty) \rightarrow \mathbb{R}$ be continuously differentiable in x , locally Lipschitz in t , and regular such that

$$W_1(x) \leq V(x, t) \leq W_2(x) \quad (8)$$

$$\dot{\tilde{V}}(x, t) \leq -W(x) \quad (9)$$

$\forall t \geq 0$, $\forall x \in \mathcal{D}$ where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions, and $W(x)$ is a continuous positive semi-definite function on \mathcal{D} . Choose $r > 0$ and $c > 0$ such that $B_r \subset \mathcal{D}$ and $c < \min_{\|x\|=r} W_1(x)$. Then, all Filippov solutions of (1) such that $x(t_0) \in \{x \in B_r \mid W_2(x) \leq c\}$ are bounded and satisfy

$$W(x(t)) \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (10)$$

Proof: Let $x(t)$ be any arbitrary Filippov solution of (1).^{a.e.}

Then, from Lemma 1, and (9), $\dot{\tilde{V}}(x(t), t) \leq -W(x(t))$, which is the condition (4). Since the selection of $x(t)$ is arbitrary, Corollary 1 can be used to imply that the result in (7) holds for each $x(t)$. Hence, Corollary 2 holds. ■

Remark 3. In the case of some systems (e.g., closed loop error systems with sliding mode control laws), it may be possible to show that Corollary 2 is more easily applied. However, in other cases, it may be difficult to satisfy the inequality in (9). The usefulness of Corollary 1 is demonstrated in those cases where it is difficult or impossible to show that the inequality in (9) can be satisfied, but it is possible to show that (4) can be satisfied for almost all time.

IV. CONCLUSION

In this note, the Lasalle-Yoshizawa Theorem is extended to differential systems whose right-hand sides are discontinuous in the state and piecewise continuous in time. The result presents two theoretical tools applicable to nonautonomous systems with discontinuities in the closed-loop error system. Generalized Lyapunov-based analysis methods are developed utilizing differential inclusions in the sense of Filippov to achieve asymptotic convergence when the candidate Lyapunov derivative is upper bounded by a negative semi-definite function.

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